12 A discrete random variable X has the Poisson distribution given by

$$P(X = r) = e^{-a}\frac{a^r}{r!}, \quad r = 0, 1, 2, \ldots .$$

Prove that the mean and the variance of X are each equal to a.

When a trainee typist types a document the number of mistakes made on any one page is a Poisson variable with mean 3, independently of the number of mistakes made on any other page. Use tables, or otherwise, to find, to three significant figures,

(i) the probability that the number of mistakes on the first page is less than two,

(ii) the probability that the number of mistakes on the first page is more than four.

Find expressions in terms of e for

(iii) the probability that the first mistake appears on the second page,

(iv) the probability that the first mistake appears on the second page and the second mistake appears on the third page.

Evaluate these expressions, giving your answers to four significant figures.

When the typist types a 48-page document the total number of mistakes made by the typist is a Poisson variable with mean 144. Use a suitable approximate method to find, to three decimal places, the probability that this total number of mistakes is greater than 130. **14**

Total 116

Joint Matriculation Board

General Certificate of Education

Mathematics

(Pure and Applied Mathematics)

Special Paper

Tuesday 26 June 1984 9.30-12.30

Careless work and untidy work will be penalised.

There are 11 questions on this paper.

There is no restriction on the number of questions which you may attempt nor on the order in which you attempt them. You are not required to answer all the questions.

The marks allocated to questions are given in the right-hand margin of the question paper.

A booklet of definitions, formulae and statistical tables is supplied.

The use of mathematical tables, calculators and slide rules is permitted.

1 A particle of unit mass moves under the action of n forces directed towards n fixed points $A_1, A_2, \ldots, A_n$, respectively. The force directed towards A_i is of magnitude k_i times the distance of the particle from A_i, where k_i is a constant, $i = 1, 2, \ldots, n$. When the particle is at a point B its acceleration is zero. When it is at another point C which is at a distance d from B, its acceleration is f. Find the magnitude of f in terms of $k_1, k_2, \ldots, k_n$ and d. **6**

2 A box contains n balls, of which two are white. Balls are to be drawn at random from the box, one after another without replacement, until both white balls have been drawn. Let X denote the number of balls that will be drawn. Show that

$$P(X = r) = \frac{2(r-1)}{n(n-1)}, \qquad r = 2, 3, \ldots, n.$$

Find, in as simple a form as possible, an expression for the mean value of X. **8**

3 Prove that for all real values of k except zero the equation

$$kx^2 + (2 + k)x - (1 + k) = 0$$

has two real distinct roots.

Find the range of values of k for which both roots are positive. **8**

4 Two variable complex numbers are denoted by $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

(i) Given that $x_1 + 2y_1 = 2$, $y_1 \neq 1$, and that

$$z_2 = \frac{z_1 - 2}{z_1 - i},$$

prove that z_2 is real.

(ii) Given that $|z_1| = 1$, $x_1 \leqslant 0$, and that

$$z_2 = \frac{2}{z_1 - 1},$$

prove that x_2 is constant. Also, by considering the possible positions of the point which represents z_2 on an Argand diagram, or otherwise, find the sets of possible values of $|z_2|$ and of $\arg z_2$. **9**

5 In the tetrahedron $OABC$, the points H and K are the mid-points of the edges OB and AC respectively. Given that

$$\overrightarrow{OA} = \mathbf{a},\ \overrightarrow{OB} = \mathbf{b} \text{ and } \overrightarrow{BC} = \mathbf{p},$$

express $\overrightarrow{AC}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{p}$, and show that

$$\overrightarrow{HK} = \tfrac{1}{2}(\mathbf{a} + \mathbf{p}).$$

Given also that OB is perpendicular to OA and to BC, and that $OA = BC$, show that AC is perpendicular to HK. Hence, or otherwise, show that $\angle OAC = \angle ACB$. **9**

6 The continuous random variable X has probability density function f given by

$$\begin{aligned} f(x) &= a &&, \quad 0 < x < 1 \\ f(x) &= b(4-x) &&, \quad 1 \leqslant x \leqslant 4 \\ f(x) &= 0 &&, \quad \text{otherwise,} \end{aligned}$$

where a and b are constants. Given that the mean of the distribution is $\frac{7}{5}$, show that a and b must have the values $\frac{2}{5}$ and $\frac{2}{15}$, respectively.

Sketch the graph of $f(x)$, and determine the median value of X, giving your answer correct to two decimal places. **9**

7 Seeds of a certain kind are planted in plastic strips, each strip containing 20 seeds. The strips are arranged on trays, each tray containing six strips, and then left to germinate. The seeds germinate independently, the probability of any particular seed germinating being 0.9. Show that the probability that in any one tray at least five of the six strips will each contain more than 17 germinated seeds is 0.372 correct to three decimal places.

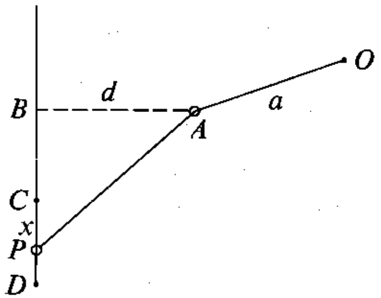
Fifty randomly selected trays are examined. Find an approximate value for the probability that at least 25 of the 50 trays have at least five strips each containing more than 17 germinated seeds, giving your answer to two decimal places. **10**

8 A particle of mass m is attached to one end A of a light inextensible string and the other end of the string is attached to a fixed point O. The particle moves in a vertical circle of centre O, and its speeds at the highest and lowest points of the circle are u and ku, respectively. Show that, when OA makes an angle θ with the downward vertical, the tension in the string is given by

$$2\left[\frac{k^2+1}{k^2-1} + \frac{3}{2}\cos\theta\right] mg.$$

Given that the string cannot support a tension greater than $9mg$, find the range of possible values of k. **12**

9



The diagram shows a smooth ring P of mass m threaded onto a fixed smooth vertical wire. Another smooth ring, A, is fixed at a perpendicular distance d from the wire. A light elastic string, of modulus λ and natural length a, is threaded through A and has one end attached to P and the other end to a fixed point O at a distance a from A. The point B lies on the wire at the same horizontal level as A. The ring P can rest in equilibrium at the point C on the wire. Show that the distance CB does not depend on d.

The ring P is pulled down to a point D at a distance b below C and is released from rest. Find an expression for the acceleration of P when its displacement is x below C, and show that the time that elapses before P returns to D is

$$2\pi(ma/\lambda)^{\frac{1}{2}}.$$

In the case when $b = 2mga/\lambda$, show that P will rise above B and find

(i) the time that elapses from the instant of release before P first reaches B,

(ii) the work that has been done by the resultant force acting on the ring P during its motion from D to B. 14

10 Given that f is a function with the property that, for any non-zero constant a,

$$f(ax) = \frac{1}{a} f(x),$$

show, by using the substitution $x = au$, that

$$\int_a^{ab} f(x)\,dx = \int_1^b f(x)\,dx.$$

Hence, or otherwise, defining ln a as $\int_1^a \frac{1}{x}\,dx$, where $a > 0$, show that

$$\ln a + \ln b = \ln ab.$$

Deduce that

$$\ln \frac{1}{a} = -\ln a.$$

The number e is defined by the relation $\ln e = 1$. By interpreting

$$\int_1^e \frac{1}{x}\,dx$$

as an area, prove that $e > 2$.

Evaluate

$$\int_0^1 \frac{4}{4-u^2}\,du$$

and hence, or otherwise, prove also that $e < 3$. 16

11 The normal at the point $P(ap^2, 2ap)$ on the parabola $y^2 = 4ax$ intersects the parabola again at Q and the tangents at P and Q intersect at R. Show that the coordinates of R are

$$\left[-a(2+p^2),\ \frac{-2a}{p}\right],$$

and find the Cartesian equation of the locus, C, of R as p varies.

The tangent at the point $T\,[-a(2+t^2), -2a/t]$ on the locus C intersects C again at the point S. Find, in terms of t, the coordinates of S. Given that the tangent to C at the point T is also normal to C at S, find the two possible values of t. **16**

Total 117

Joint Matriculation Board

General Certificate of Education

Mathematics

(Pure Mathematics)

Special Paper

Tuesday 26 June 1984 9.30-12.30

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A booklet of definitions, formulae and statistical tables is supplied.

The use of mathematical tables, calculators and slide rules is permitted.

1 Prove that for all real values of k except zero the equation

$$kx^2 + (2 + k)x - (1 + k) = 0$$

has two real distinct roots.

Find the range of values of k for which both roots are positive. **8**

2 The complex numbers α, β, γ satisfy the equations

$$\alpha + \beta + \gamma = 1,$$
$$\alpha^2 + \beta^2 + \gamma^2 = -3,$$
$$\alpha^4 + \beta^4 + \gamma^4 = 5.$$

Prove that $\alpha\beta\gamma = 1$ and find $\alpha^3 + \beta^3 + \gamma^3$. **8**

3 Two variable complex numbers are denoted by $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$.

(i) Given that $x_1 + 2y_1 = 2$, $y_1 \neq 1$, and that

$$z_2 = \frac{z_1 - 2}{z_1 - \mathrm{i}},$$

prove that z_2 is real.

(ii) Given that $|z_1| = 1$, $x_1 \leqslant 0$, and that

$$z_2 = \frac{2}{z_1 - 1},$$

prove that x_2 is constant. Also, by considering the possible positions of the point which represents z_2 on an Argand diagram, or otherwise, find the sets of possible values of $|z_2|$ and of $\arg z_2$. **9**

4 In the tetrahedron $OABC$, the points H and K are the mid-points of the edges OB and AC respectively. Given that

$$\overrightarrow{OA} = \mathbf{a},\ \overrightarrow{OB} = \mathbf{b} \text{ and } \overrightarrow{BC} = \mathbf{p},$$

express $\overrightarrow{AC}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{p}$, and show that

$$\overrightarrow{HK} = \tfrac{1}{2}(\mathbf{a} + \mathbf{p}).$$

Given also that OB is perpendicular to OA and to BC, and that $OA = BC$, show that AC is perpendicular to HK. Hence, or otherwise, show that $\angle OAC = \angle ACB$. **9**

5 A finite group G is of even order. By considering the inverses of the elements of G, or otherwise, show that there cannot be an even number of elements of order 2.

Part of the composition table of a group of order 6 is shown below. Determine which of the elements of the group is represented by x and which by y.

Show that $z \neq d$ and hence find z. **11**

	a	b	c	d	e	f
a		e				
b	y					
c			e			
d				e		
e	a	b	c	d	e	f
f			z	b		x

6 A linear transformation with matrix M maps the point P (3,6,6) on to the point Q (1,4,8). Given that the transformation is a reflection in a plane, find the equation of the plane. Find also the matrix M.

Another transformation also maps P on to Q. This transformation is a rotation about a line L through the origin and the direction of L is such that the angle of the rotation is the least possible. Find the direction of L. **13**

7 The region in the x–y plane defined by $1 \leqslant x \leqslant a$, $0 \leqslant y \leqslant x^{-n}$ is rotated through 2π radians about the x-axis to form a solid of revolution having a volume V. Given that $a > 1$ and that $n > \frac{1}{2}$, find the limit of V as $a \to \infty$.

In the case $n = 1$, the area of the *curved* surface of the above solid is denoted by S. Express S as an integral with respect to x. By using the substitution $x^2 = \sinh u$, or otherwise, evaluate the integral in terms of a and hence show that $S \to \infty$ as $a \to \infty$. **14**

8 A target T moves in the x-y plane along the line $x = k$, where $k > 0$. It starts at the point $(k, 0)$ at time $t = 0$ and moves in the positive y-direction with constant speed u.

A pursuer P is at the origin at $t = 0$ and moves with a velocity which is always directed towards T and which has a constant component u in the x-direction. Denoting the coordinates of P at time t by (x,y), show that

$$(k-x)\frac{dy}{dx} + y = x.$$

By putting $k - x = p$ and $x - y = q$, or otherwise, find y in terms of x.

Show that, when the speed of P is λu,

$$x = k\left(1 - e^{-\sqrt{(\lambda^2-1)}}\right)$$

and find the distance between P and T at this instant. **15**

9 Given that f is a function with the property that, for any non-zero constant a,

$$f(ax) = \frac{1}{a} f(x),$$

show, by using the substitution $x = au$, that

$$\int_a^{ab} f(x)\,dx = \int_1^b f(x)\,dx.$$

Hence, or otherwise, defining $\ln a$ as $\int_1^a \frac{1}{x}\,dx$, where $a > 0$, show that

$$\ln a + \ln b = \ln ab.$$

Deduce that

$$\ln \frac{1}{a} = -\ln a.$$

The number e is defined by the relation $\ln e = 1$. By interpreting

$$\int_1^e \frac{1}{x}\,dx$$

as an area, prove that $e > 2$.

Evaluate
$$\int_0^1 \frac{4}{4-u^2}\,du$$

and hence, or otherwise, prove also that $e < 3$. **16**

10 The normal at the point $P(ap^2, 2ap)$ on the parabola $y^2 = 4ax$ intersects the parabola again at Q and the tangents at P and Q intersect at R. Show that the coordinates of R are

$$\left[-a(2+p^2),\ \frac{-2a}{p}\right],$$

and find the Cartesian equation of the locus, C, of R as p varies.

The tangent at the point $T\,[-a(2+t^2), -2a/t]$ on the locus C intersects C again at the point S. Find, in terms of t, the coordinates of S. Given that the tangent to C at the point T is also normal to C at S, find the two possible values of t. **16**

Total 119

Joint Matriculation Board

General Certificate of Education

Mathematics

(Pure Mathematics with Mechanics)

Special Paper

Tuesday 26 June 1984 9.30–12.30

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A booklet of definitions, formulae and statistical tables is supplied.

The use of mathematical tables, calculators and slide rules is permitted.

1 A ball is projected with speed u and at an angle of elevation α from a fixed point O. The ball moves under gravity towards a smooth vertical wall which is at a distance d from O, and after impact with the wall it returns directly to O. The coefficient of restitution between the ball and the wall is e. By equating two distinct expressions for the total time of flight, or otherwise, show that

$$eu^2 \sin 2\alpha = (1 + e) gd.$$ 7

2 Prove that for all real values of k except zero the equation

$$kx^2 + (2 + k)x - (1 + k) = 0$$

has two real distinct roots.

Find the range of values of k for which both roots are positive. 8

3 A smooth ring P of mass m is threaded onto a fixed smooth vertical wire. Another smooth ring Q is fixed at a perpendicular distance d from the wire. A light elastic string, of modulus λ and natural length a, is threaded through Q and has one end attached to P and the other end to a fixed point O at a distance a from Q. Show that the distance of the equilibrium position of P below the level of Q is independent of d.

The ring P is pulled down to a point at a distance b below the level of Q and released from rest. Show that the speed of P in the subsequent motion is independent of d, and find the least value of b for which P reaches the level of Q. 8

4 Two variable complex numbers are denoted by $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

(i) Given that $x_1 + 2y_1 = 2, y_1 \neq 1$, and that

$$z_2 = \frac{z_1 - 2}{z_1 - i},$$

prove that z_2 is real.

(ii) Given that $|z_1| = 1, x_1 \leqslant 0$, and that

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prove that x_2 is constant. Also, by considering the possible positions of the point which represents z_2 on an Argand diagram, or otherwise, find the sets of possible values of $|z_2|$ and of $\arg z_2$. 9

5 In the tetrahedron $OABC$, the points H and K are the mid-points of the edges OB and AC respectively. Given that

$$\overrightarrow{OA} = \mathbf{a}, \overrightarrow{OB} = \mathbf{b} \text{ and } \overrightarrow{BC} = \mathbf{p},$$

express $\overrightarrow{AC}$ in terms of **a**, **b** and **p**, and show that

$$\overrightarrow{HK} = \tfrac{1}{2}(\mathbf{a} + \mathbf{p}).$$

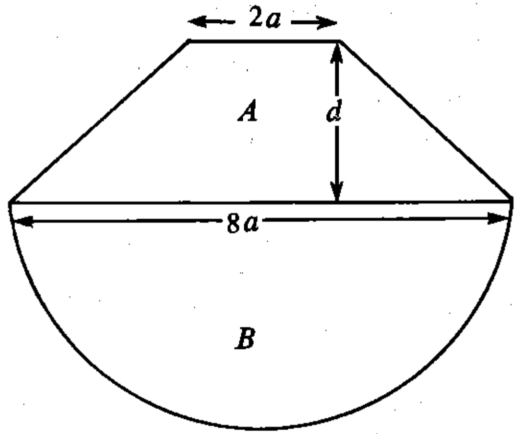
Given also that OB is perpendicular to OA and to BC, and that $OA = BC$, show that AC is perpendicular to HK. Hence, or otherwise, show that $\angle OAC = \angle ACB$. 9

6 A particle of mass m is attached to one end A of a light inextensible string and the other end of the string is attached to a fixed point O. The particle moves in a vertical circle of centre O, and its speeds at the highest and lowest points of the circle are u and ku, respectively. Show that, when OA makes an angle θ with the downward vertical, the tension in the string is given by

$$2\left[\frac{k^2+1}{k^2-1}+\tfrac{3}{2}\cos\theta\right]mg.$$

Given that the string cannot support a tension greater than $9mg$, find the range of possible values of k. 12

7



The diagram shows a cross-section of a solid composite body consisting of a frustum A of a right circular cone and a hemisphere B with the larger plane face of the frustum and the plane face of the hemisphere coincident. The radii of the plane ends of the frustum are a and $4a$, respectively, and the radius of the hemisphere is $4a$. The frustum has a uniform density 16ρ and the hemisphere a uniform density ρ. The plane faces of the frustum are at a distance d apart, and it is *given* that the centre of mass of the frustum is at a perpendicular distance $9d/28$ from its larger plane face. When the composite body is placed with any point of its hemispherical surface on a horizontal plane it rests in equilibrium in that position. Show that

$$d = 4a/3.$$

The body is placed on a smooth plane, which is inclined at an angle θ to the horizontal. A light string has one end attached to a point on the circumference of the smaller plane face of the frustum and the other end attached to a point O on the plane. The body, which is of weight W, rests in equilibrium with the plane face of the hemisphere perpendicular to the plane. Show that the length of the string is $5a$. Find, in terms of θ and W, the tension in the string. 15

8 When a particle of mass m is moving through air with speed v it experiences a resistance mkv^2, where k is a positive constant. The particle is held at a point O and then released from rest. Show that the speed of the falling particle will never exceed $(g/k)^{\frac{1}{2}}$.

Find the work that has been done against the air resistance when the particle has fallen to a point A at a distance h vertically below O.

When the particle reaches A it is subjected to a vertical upward impulse of just sufficient magnitude to cause the particle to return to O. Find, in terms of m, g, k and h, the magnitude of this impulse. **15**

9 Given that f is a function with the property that, for any non-zero constant a,

$$f(ax) = \frac{1}{a} f(x),$$

show, by using the substitution $x = au$, that

$$\int_a^{ab} f(x)\,dx = \int_1^b f(x)\,dx.$$

Hence, or otherwise, defining ln a as $\int_1^a \frac{1}{x}\,dx$, where $a > 0$, show that

$$\ln a + \ln b = \ln ab.$$

Deduce that

$$\ln \frac{1}{a} = -\ln a.$$

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Evaluate

$$\int_0^1 \frac{4}{4-u^2}\,du$$

and hence, or otherwise, prove also that e < 3. **16**

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and find the Cartesian equation of the locus, C, of R as p varies.

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16

Total 115

Joint Matriculation Board

General Certificate of Education

Mathematics

(Pure Mathematics with Statistics)

Special Paper

Tuesday 26 June 1984 9.30–12.30

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1 A box contains n balls, of which two are white. Balls are to be drawn at random from the box, one after another without replacement, until both white balls have been drawn. Let X denote the number of balls that will be drawn. Show that

$$P(X = r) = \frac{2(r-1)}{n(n-1)}, \qquad r = 2,3,\ldots,n.$$

Find, in as simple a form as possible, an expression for the mean value of X. 8

2 Prove that for all real values of k except zero the equation

$$kx^2 + (2 + k)x - (1 + k) = 0$$

has two real distinct roots.

Find the range of values of k for which both roots are positive. 8

3 The numbers of emissions in a period of one minute from two radioactive sources may be assumed to be independent random variables, each having a Poisson distribution with mean μ. A Geiger counter is used to record the combined number of emissions from the two sources in a period of one minute.

(i) Find, in terms of μ, the probability that there will be at least one emission recorded during the period.

(ii) Show that the probability that two emissions will be recorded during the period is *exactly* twice the probability that there will be one emission from each of the two sources during the period.

(iii) Given that at least one emission occurred during the period, show that the probability that all the emissions came from the same source is

$$2/(1 + e^{\mu}).$$ 9

4 Two variable complex numbers are denoted by $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.

(i) Given that $x_1 + 2y_1 = 2$, $y_1 \neq 1$, and that

$$z_2 = \frac{z_1 - 2}{z_1 - i},$$

prove that z_2 is real.

(ii) Given that $|z_1| = 1$, $x_1 \leqslant 0$, and that

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5 In the tetrahedron $OABC$, the points H and K are the mid-points of the edges OB and AC respectively. Given that

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express $\overrightarrow{AC}$ in terms of **a**, **b** and **p**, and show that

$$\overrightarrow{HK} = \tfrac{1}{2}(\mathbf{a} + \mathbf{p}).$$

Given also that OB is perpendicular to OA and to BC, and that $OA = BC$, show that AC is perpendicular to HK. Hence, or otherwise, show that $\angle OAC = \angle ACB$. 9

6 The random variables X and Y are known to be normally distributed with means μ and λ, and variances 4 and 16, respectively. A test is required of the null hypothesis that $\mu = \lambda$ against the alternative hypothesis that $\mu \neq \lambda$. The test is to be based on the means of independent random samples of 16 observations of X and 25 observations of Y.

(i) One possible test is to reject the null hypothesis only if the *numerical* difference between the two sample means exceeds a certain value c. Obtain the value of c for this test to have a significance level of 0.05.

(ii) Another possible test is to reject the null hypothesis only if the 95% symmetrical confidence intervals for μ and λ do not overlap. Show that the significance level of this test is approximately 0.007. **12**

7 Two variables x and y are known to be related by the formula $y = \lambda x$, where λ is an unknown constant. To obtain information on λ, experiments were performed with x having the values $x_1, x_2, \ldots, x_n$, and the corresponding values of y were measured. Denoting the measured values of y by $y_1, y_2, \ldots, y_n$, respectively, show that the least squares estimate, m, of λ is given by

$$m = \frac{\sum x_i y_i}{\sum x_i^2} \, .$$

Assuming that $y_i = \lambda x_i + \varepsilon_i$, where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are independent random errors each having mean zero and variance σ^2, show that m is an unbiased estimate of λ and that its sampling variance is

$$\frac{\sigma^2}{\sum x_i^2} \, .$$

The actual values obtained in the experiments are shown in the following table.

x_i	1	2	3	4	5
y_i	2.6	5.1	7.7	9.8	12.7

(i) Calculate the value of m.

(ii) Assuming that the sampling distribution of m is normal and that $\sigma = 0.2$, determine the limits of the 95% symmetrical confidence intervals for λ and for the true value of y when $x = 3$. **13**

8 The continuous random variable X has the probability density function f, where

$$f(x) = \tfrac{2}{5}, \quad 0 < x < 1,$$

$$f(x) = \tfrac{2}{15}(4 - x), \quad 1 \leqslant x \leqslant 4,$$

$$f(x) = 0, \quad \text{otherwise.}$$

(i) Find the distribution function F of X. Hence, or otherwise, determine the value of the lower quartile of X and, correct to two decimal places, the value of the median of X.

(ii) Let G denote the distribution function of the random variable Y given by

$$Y = \frac{12}{X + 2}.$$

Show that, for $2 \leqslant y \leqslant 4$,

$$G(y) = \frac{12}{5}\left(1 - \frac{2}{y}\right)^2.$$

Obtain an expression for $G(y)$ for $4 < y < 6$.

Hence, or otherwise, obtain expressions for $g(y)$, where g is the probability density function of Y.

9 Given that f is a function with the property that, for any non-zero constant a,

$$f(ax) = \frac{1}{a} f(x),$$

show, by using the substitution $x = au$, that

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$$\left[-a(2 + p^2),\ \frac{-2a}{p}\right],$$

and find the Cartesian equation of the locus, C, of R as p varies.

The tangent at the point $T\,[-a(2 + t^2), -2a/t]$ on the locus C intersects C again at the point S. Find, in terms of t, the coordinates of S. Given that the tangent to C at the point T is also normal to C at S, find the two possible values of t. **16**

Total 116

Joint Matriculation Board

General Certificate of Education

Mathematics

(Pure and Applied Mathematics)

Special Paper

Tuesday 25 June 1985 9.30-12.30

Careless work and untidy work will be penalised.

There are 11 questions on this paper.

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The marks allocated to questions are given in the right-hand margin of the question paper.

A booklet of definitions, formulae and statistical tables is supplied.

The use of mathematical tables, calculators and slide rules is permitted.

1 Given that

$$f(\theta) = 1 + \cos\theta + i\sin\theta,$$

show that

$$f\left(\theta + \frac{\pi}{2}\right) = 1 - \sin\theta + i\cos\theta.$$

The complex numbers z_1 and z_2 are given by

$$z_1 = 1 + \cos\theta + i\sin\theta,$$
$$z_2 = 1 - \sin\theta + i\cos\theta.$$

Show that z_1 is a real multiple of $\cos\theta/2 + i\sin\theta/2$.

Given that $0 < \theta < \pi/2$, find arg z_1 and arg z_2, and show that

$$\arg(z_1 z_2) = \theta + \frac{\pi}{4}\,.$$

Find $\arg(z_1 z_2)$ when $\frac{\pi}{2} < \theta < \pi$. 7

2 A light elastic string is stretched between two fixed points A and B on a smooth horizontal table. The string is of natural length a and $AB = b$, where $b > a$. A particle P is attached to the string at the point which is initially at a distance kb from A. The particle is displaced and then released so that it oscillates on the line AB. Throughout the motion the portions AP and PB of the string are both in tension. Show that the motion is simple harmonic.

Show also that the periods of oscillation when $k = \frac{1}{2}$ and when $k = \frac{1}{3}$ are in the ratio $3 : 2\sqrt{2}$. 7

3 From a point O two particles are projected simultaneously with speeds u and v and at acute angles α and β, respectively, to the horizontal. The particles move freely under gravity and each particle subsequently passes through a point A. Show that the time interval between the passing of the particles through A is

$$\frac{2uv\sin|\alpha - \beta|}{g(u\cos\alpha + v\cos\beta)}\,.$$ 7

4 The function f is defined for all real x by

$$f(x) = x^3 - 3x - 2 - 27p.$$

Show that, when $p > 0$, the equation $f(x) = 0$ has just one real root, α, and that $2 < \alpha < 2 + 3p$.

Let x_0 denote any first approximation to α such that $f(x_0) > 0$. Let $x_1, x_2, \ldots, x_n$ denote successive further approximations obtained by using Newton's method. Show, by graphical considerations or otherwise, that

$$\alpha < x_n < x_{n-1} < \ldots < x_1 < x_0.$$

Given that p is small and that $x_0 = 2 + 3p$, show that

$$x_1 \approx 2 + 3p - 6p^2.$$ 8

5 Given that

$$\frac{1}{(1 + a^3x^3)(1 + b^3x^3)} \equiv \frac{A}{1 + a^3x^3} + \frac{B}{1 + b^3x^3} \quad (a \neq b),$$

where a and b are constants, find A and B in terms of a and b.

The function f is defined for $x \in \mathbb{R}$, $ax \neq -1$, $bx \neq -1$, by

$$f(x) = \frac{1 + bx}{(1 + a^3x^3)(1 + b^3x^3)}\,,$$

where a and b are constants such that $a > b > 0$. Find the coefficients of x^{3r}, x^{3r+1} and x^{3r+2} in the expansion of $f(x)$ in ascending powers of x.

Determine the values of x for which this expansion is valid. 8

6 A battery used to operate a device has n cells, each of which, independently of the other cells, has probability p of being functional. When r of the cells are functional, the probability that the battery will operate the device is

$\frac{r}{n}$, for $0 \leqslant r \leqslant n$.

(i) Let X denote the number of functional cells in the battery. Name the distribution of X and write down expressions for $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$.

(ii) Obtain an expression for the joint probability that the battery will operate the device *and* have exactly r functional cells. Hence show that the probability that the battery will operate the device is p.

(iii) Given that the battery does operate the device, find an expression for the probability that the battery has exactly r functional cells. Hence, or otherwise, show that when the device is operating, the expected value of the number of functional cells in the battery is

$$1 - p + np.$$

10

7 The variables x and y satisfy the equation

$$(2 + \cos x)(2 - \cos y) = 3,$$

where $0 < x < \pi$ and $0 < y < \pi$. Express $\sin y$ in terms of $\sin x$ and $\cos x$ and show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{3}}{2 + \cos x}.$$

Evaluate

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{27}{(2 - \cos y)^3}\, \mathrm{d}y.$$

11

8 A tank with vertical faces has a horizontal rectangular base with sides of lengths X m and $(10 - X)$ m, where X is a continuous random variable having a uniform distribution between 6 and 9. Water is poured into the tank. Given that the volume of water in the tank is 100 m^3, calculate, to three decimal places, the probability that the depth of water in the tank is greater than 5 m.

Also calculate, to the nearest m^3, the least volume of water that the tank must contain for the probability to be at least 0.9 that the depth of water exceeds 5 m. **12**

9 The point O is the centre of the circle of radius R through the vertices of a triangle ABC. The position vectors of A, B and C relative to O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, and H is the point with position vector $\mathbf{a} + \mathbf{b} + \mathbf{c}$. Prove that AH, BH and CH are perpendicular to BC, CA and AB, respectively.

The mid-points of BC, CA, AB, HA, HB and HC are A', B', C', L, M and N, respectively. Show that the lines LA', MB' and NC' are concurrent and find, in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, the position vector of their point of concurrence, K. Hence show that OKH is a straight line.

Prove also that the points A', B', C', L, M and N all lie on a circle with centre K and state the radius of this circle in terms of R. **13**

10 The deceleration of an aircraft after touching down on a runway is $a + bv^2$, where v is the speed of the aircraft and a and b are positive constants. Given that it takes half the runway to halt the aircraft when its touchdown speed is U, show that the maximum touchdown speed from which the aircraft can be brought to a halt in the full length of the runway is

$$\{2 + (b/a)U^2\}^{\frac{1}{2}}\, U.$$

Given that this maximum touchdown speed is $U\sqrt{5}$, find, in terms of a and b, the time taken for the aircraft to be halted from speed U.

The components a and bv^2 of the deceleration are due to braking and air resistance, respectively. Given that the mass of the aircraft is M, find, in terms of a, b and M, the work done against the air resistance when the aircraft is brought to a halt from its maximum touchdown speed $U\sqrt{5}$. **14**

11 The function f has domain $\mathbb{R}$ and its inverse function is denoted by g.

(i) In the case when

$$f(x) = 3x + |x|,$$

find expressions for $f[f(x)]$ and $g(x)$ in the form $px + q\,|x|$, where p and q are constants.

(ii) In the case when

$$f(x) = x^3 - 3x^2 + 4x - 10,$$

show that g exists and state its domain and range. Show, by a graphical method or otherwise, that there is one and only one value of x such that

$$f(x) = g(x).$$

Prove that

$$f'[g(x)]\, g'(x) = 1.$$

Show that $g(2) = 3$, and find the values of $g'(2)$ and $g''(2)$. **20**

Total 117

Joint Matriculation Board

General Certificate of Education

Mathematics

(Pure Mathematics)

Special Paper

Tuesday 25 June 1985 9.30-12.30

Careless work and untidy work will be penalised.

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The marks allocated to questions are given in the right-hand margin of the question paper.

A booklet of definitions, formulae and statistical tables is supplied.

The use of mathematical tables, calculators and slide rules is permitted.

1 Given that

$$f(\theta) = 1 + \cos\theta + i\sin\theta,$$

show that

$$f\left(\theta + \frac{\pi}{2}\right) = 1 - \sin\theta + i\cos\theta.$$

The complex numbers z_1 and z_2 are given by

$$z_1 = 1 + \cos\theta + i\sin\theta,$$
$$z_2 = 1 - \sin\theta + i\cos\theta.$$

Show that z_1 is a real multiple of $\cos\theta/2 + i\sin\theta/2$.

Given that $0 < \theta < \pi/2$, find $\arg z_1$ and $\arg z_2$, and show that

$$\arg(z_1 z_2) = \theta + \frac{\pi}{4}.$$

Find $\arg(z_1 z_2)$ when $\frac{\pi}{2} < \theta < \pi$. **7**

2 The point P, with parameter θ, lies on the curve C defined by the equations

$$x = 4e^{\theta}\cos\theta, \quad y = 4e^{\theta}\sin\theta.$$

The unit vector, in the sense of θ increasing, along the tangent to the curve at P is **t**. The distance of P from the origin O and the arc length of the curve between O and P are denoted by r and s, respectively. Show that

$$\mathbf{t} = \frac{1}{\sqrt{2}}\left[(\cos\theta - \sin\theta)\,\mathbf{i} + (\cos\theta + \sin\theta)\mathbf{j}\right],$$

and hence, or otherwise, express $\left|\frac{d\mathbf{t}}{ds}\right|$ in terms of r. **7**

3 Find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 2e^{-x}$$

such that $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. **8**

4 Given that

$$\frac{1}{(1 + a^3x^3)(1 + b^3x^3)} \equiv \frac{A}{1 + a^3x^3} + \frac{B}{1 + b^3x^3} \quad (a \neq b),$$

where a and b are constants, find A and B in terms of a and b.

The function f is defined for $x \in \mathbb{R}$, $ax \neq -1$, $bx \neq -1$, by

$$f(x) = \frac{1 + bx}{(1 + a^3x^3)(1 + b^3x^3)},$$

where a and b are constants such that $a > b > 0$. Find the coefficients of x^{3r}, x^{3r+1} and x^{3r+2} in the expansion of $f(x)$ in ascending powers of x.

Determine the values of x for which this expansion is valid. **8**

5 Given that r and C are constants, show that

$$\int e^{-x}\cos\left(x + r\frac{\pi}{4}\right)dx = \frac{-e^{-x}}{2^{\frac{1}{2}}}\cos\left[x + (r+1)\frac{\pi}{4}\right] + C.$$

Given that n is a positive integer, show that

$$\int_0^{\infty} x^n e^{-x}\cos x\,dx = \frac{n!}{2^{\frac{(n+1)}{2}}}\cos\left[(n+1)\frac{\pi}{4}\right].$$ **10**

6 The variables x and y satisfy the equation

$$(2 + \cos x)(2 - \cos y) = 3,$$

where $0 < x < \pi$ and $0 < y < \pi$. Express $\sin y$ in terms of $\sin x$ and $\cos x$ and show that

$$\frac{dy}{dx} = \frac{\sqrt{3}}{2 + \cos x}.$$

Evaluate

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{27}{(2 - \cos y)^3}\,dy.$$ **11**

7 The point O is the centre of the circle of radius R through the vertices of a triangle ABC. The position vectors of A, B and C relative to O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, respectively, and H is the point with position vector $\mathbf{a} + \mathbf{b} + \mathbf{c}$. Prove that AH, BH and CH are perpendicular to BC, CA and AB, respectively.

The mid-points of BC, CA, AB, HA, HB and HC are A', B', C', L, M and N, respectively. Show that the lines LA', MB' and NC' are concurrent and find, in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, the position vector of their point of concurrence, K. Hence show that OKH is a straight line.

Prove also that the points A', B', C', L, M and N all lie on a circle with centre K and state the radius of this circle in terms of R. **13**

8 The equation of a curve C is

$$y = \frac{x + a}{x^2 + bx + c^2},$$

where a, b and c are constants. For points (x,y) on C

(i) find the condition which must be satisfied by the constants so that y lies in a finite interval [this interval need not be specified];

(ii) find the two conditions which must be satisfied by the constants so that y can only take values lying outside a finite interval [this interval need not be specified];

(iii) find the conditions that must be satisfied by the constants so that y takes all real values. **15**

9 A group G is of order 8 and the binary operation is multiplication.

(i) By considering the inverses of the elements, or otherwise, prove that G contains at least one element of order 2.

(ii) By considering $(ab)^2$, where a and b are elements of G, or otherwise, prove that if every element of G, other than the identity, is of order 2 then G is commutative.

(iii) Given that the order of any finite group is exactly divisible by the order of each of its elements, prove that, if G is not commutative, it contains at least one element of order 4.

(iv) The cyclic subgroup of G generated by an element α of G of order 4 is denoted by G_1 and β denotes an element of G not in G_1. Express the other elements of G, which are not in G_1, in terms of α and β, and show that β^2 is an element of G_1. **17**

10 The function f has domain $\mathbb{R}$ and its inverse function is denoted by g.

(i) In the case when

$$f(x) = 3x + |x|,$$

find expressions for $f[f(x)]$ and $g(x)$ in the form $px + q\,|x|$, where p and q are constants.

(ii) In the case when

$$f(x) = x^3 - 3x^2 + 4x - 10,$$

show that g exists and state its domain and range. Show, by a graphical method or otherwise, that there is one and only one value of x such that

$$f(x) = g(x).$$

Prove that

$$f'[g(x)]\,g'(x) = 1.$$

Show that $g(2) = 3$, and find the values of $g'(2)$ and $g''(2)$. **20**

Total 116

Joint Matriculation Board

General Certificate of Education

Mathematics
(Pure Mathematics with Mechanics)
Special Paper

Tuesday 25 June 1985 9.30-12.30

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$$f(\theta) = 1 + \cos\theta + i\sin\theta,$$

show that

$$f\left(\theta + \frac{\pi}{2}\right) = 1 - \sin\theta + i\cos\theta.$$

The complex numbers z_1 and z_2 are given by

$$z_1 = 1 + \cos\theta + i\sin\theta,$$
$$z_2 = 1 - \sin\theta + i\cos\theta.$$

Show that z_1 is a real multiple of $\cos\theta/2 + i\sin\theta/2$.

Given that $0 < \theta < \pi/2$, find $\arg z_1$ and $\arg z_2$, and show that

$$\arg(z_1 z_2) = \theta + \frac{\pi}{4}.$$

Find $\arg(z_1 z_2)$ when $\frac{\pi}{2} < \theta < \pi$. 7

2 From a point O two particles are projected simultaneously with speeds u and v and at acute angles α and β, respectively, to the horizontal. The particles move freely under gravity and each particle subsequently passes through a point A. Show that the time interval between the passing of the particles through A is

$$\frac{2uv\sin|\alpha - \beta|}{g(u\cos\alpha + v\cos\beta)}.$$ 7

3 Given that

$$\frac{1}{(1 + a^3x^3)(1 + b^3x^3)} \equiv \frac{A}{1 + a^3x^3} + \frac{B}{1 + b^3x^3} \quad (a \neq b),$$

where a and b are constants, find A and B in terms of a and b.

The function f is defined for $x \in \mathbb{R}$, $ax \neq -1$, $bx \neq -1$, by

$$f(x) = \frac{1 + bx}{(1 + a^3x^3)(1 + b^3x^3)},$$

where a and b are constants such that $a > b > 0$. Find the coefficients of x^{3r}, x^{3r+1} and x^{3r+2} in the expansion of $f(x)$ in ascending powers of x.

Determine the values of x for which this expansion is valid. 8

4 A uniform rod AB, of mass m and length $2a$, rests in a horizontal position with its end A against a rough vertical wall. The rod is supported by an inextensible string of length l which has one end attached to a point on the rod and the other end to a point on the wall vertically above A. The string is inclined at an angle α to the horizontal. A particle of mass m is attached to the rod at a distance b from A and the system is in equilibrium. Given that the coefficient of friction between the rod and the wall is $\frac{1}{4}\tan\alpha$, show that

$$\tfrac{3}{8}(a + b) \leqslant l\cos\alpha \leqslant \tfrac{5}{8}(a + b).$$ 9

5 The variables x and y satisfy the equation

$$(2 + \cos x)(2 - \cos y) = 3,$$

where $0 < x < \pi$ and $0 < y < \pi$. Express $\sin y$ in terms of $\sin x$ and $\cos x$ and show that

$$\frac{dy}{dx} = \frac{\sqrt{3}}{2 + \cos x}.$$

Evaluate

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{27}{(2 - \cos y)^3}\, dy.$$ 11

6 A smooth wire in the form of a circle is fixed in a vertical plane. Two small beads of masses m and $3m$ are threaded on the wire and are simultaneously released from rest at opposite ends of the horizontal diameter of the circle. The coefficient of restitution between the beads is e. Prove that, after the first collision and before the second collision, the greatest heights above the lowest point of the wire attained by the beads are in the ratio

$$(1 + 3e)^2 : (1 - e)^2,$$

provided that $e \leqslant \frac{1}{3}(\sqrt{8} - 1)$.

Explain what happens when $e > \frac{1}{3}(\sqrt{8} - 1)$ and find the ratio of the greatest heights in this case.

[It may be assumed that both beads have reached their greatest heights *before* the second collision.] 12

Joint Matriculation Board

General Certificate of Education

Mathematics
(Pure Mathematics with Statistics)
Special Paper

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The use of mathematical tables, calculators and slide rules is permitted.

1 A process extracts two chemicals A and B from 1 kg batches of raw material. The amount of A extracted from a batch is X grams, where X is a continuous random variable having mean μ and standard deviation 2, and the amount of B extracted is Y grams, where Y is a continuous random variable, independent of X, having mean λ and standard deviation 3. Chemical A is valued at £2 per gram, while chemical B is valued at £1 per gram. The process was applied to 100 batches, the means of the amounts of A and B extracted per batch being 5.2 grams and 8.6 grams, respectively.

Calculate approximate 95% confidence limits for the true mean combined value of the chemicals extracted per batch. 6

2 Given that

$$f(\theta) = 1 + \cos\theta + i\sin\theta,$$

show that

$$f\left(\theta + \frac{\pi}{2}\right) = 1 - \sin\theta + i\cos\theta.$$

The complex numbers z_1 and z_2 are given by

$$z_1 = 1 + \cos\theta + i\sin\theta,$$
$$z_2 = 1 - \sin\theta + i\cos\theta.$$

Show that z_1 is a real multiple of $\cos\theta/2 + i\sin\theta/2$.

Given that $0 < \theta < \pi/2$, find arg z_1 and arg z_2, and show that

$$\arg(z_1 z_2) = \theta + \frac{\pi}{4}.$$

Find $\arg(z_1 z_2)$ when $\frac{\pi}{2} < \theta < \pi$. 7

3 The events A, B and C are such that

$$P(A) = \frac{3}{5}, P(C) = \frac{1}{2}, P(B|C) = \frac{2}{5}, P(B|C') = \frac{1}{5},$$
$$P(B|A) = \frac{1}{5}, \text{ and } P(A \cap B \cap C) = \frac{3}{25}.$$

(i) Show that A and B are not independent.

(ii) Show that when both A and B occur together, C also occurs.

(iii) Show that $P(A \cap C) \geqslant \frac{1}{5}$. 8

4 Two normal distributions have means μ and λ, and have common standard deviation 5. The null hypothesis

$$H_0 : \mu - \lambda = -2$$

is to be tested against the alternative hypothesis

$$H_1 : \mu - \lambda = 2.$$

For this purpose it is decided to take independent random samples of size n from each of the two distributions and to reject H_0 only if $\bar{x} > \bar{y}$, where $\bar{x}$ and $\bar{y}$ are the means of the samples from the distributions having means μ and λ, respectively. Show that for this decision rule, the probabilities of a type-1 error and a type-2 error are equal.

Find the smallest value of n for each of these error probabilities to be less than 0.05. 8

5 Given that

$$\frac{1}{(1 + a^3x^3)(1 + b^3x^3)} \equiv \frac{A}{1 + a^3x^3} + \frac{B}{1 + b^3x^3} \quad (a \neq b),$$

where a and b are constants, find A and B in terms of a and b.

The function f is defined for $x \in \mathbb{R}$, $ax \neq -1$, $bx \neq -1$, by

$$f(x) = \frac{1 + bx}{(1 + a^3x^3)(1 + b^3x^3)},$$

where a and b are constants such that $a > b > 0$. Find the coëfficients of x^{3r}, x^{3r+1} and x^{3r+2} in the expansion of f(x) in ascending powers of x.

Determine the values of x for which this expansion is valid. 8

6 A battery used to operate a device has n cells, each of which, independently of the other cells, has probability p of being functional. When r of the cells are functional, the probability that the battery will operate the device is

$\frac{r}{n}$, for $0 \leqslant r \leqslant n$.

(i) Let X denote the number of functional cells in the battery. Name the distribution of X and write down expressions for $E(X)$ and $E(X^2)$.

(ii) Obtain an expression for the joint probability that the battery will operate the device *and* have exactly r functional cells. Hence show that the probability that the battery will operate the device is p.

(iii) Given that the battery does operate the device, find an expression for the probability that the battery has exactly r functional cells. Hence, or otherwise, show that when the device is operating, the expected value of the number of functional cells in the battery is

$$1 - p + np.$$ 10

7 The probability density function f of the continuous random variable X is such that

$$f(x) = \alpha x^{-(\alpha+1)},\ x \geqslant 1,$$
$$f(x) = 0, \text{ otherwise,}$$

where α is an unknown positive constant.

(i) Determine the distribution function of X. For the case when $\alpha = 2$, find the median value of X.

(ii) To test the null hypothesis that $\alpha = 2$ against the alternative hypothesis that $\alpha \neq 2$, it is decided to take a random sample of 100 observations of X and to count the number, R, that have values greater than 5. Given that the chosen significance level is 5% and that the observed value of R is 9, use an appropriate approximation to the sampling distribution of R when $\alpha = 2$ to carry out the test. State whether your conclusion is that $\alpha = 2$, $\alpha < 2$ or $\alpha > 2$. 10

8 The variables x and y satisfy the equation

$$(2 + \cos x)(2 - \cos y) = 3,$$

where $0 < x < \pi$ and $0 < y < \pi$. Express $\sin y$ in terms of $\sin x$ and $\cos x$ and show that

$$\frac{dy}{dx} = \frac{\sqrt{3}}{2 + \cos x}.$$

Evaluate

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{27}{(2 - \cos y)^3}\, dy.$$ 11

9 The point O is the centre of the circle of radius R through the vertices of a triangle ABC. The position vectors of A, B and C relative to O are **a**, **b** and **c**, respectively, and H is the point with position vector **a** + **b** + **c**. Prove that AH, BH and CH are perpendicular to BC, CA and AB, respectively.

The mid-points of BC, CA, AB, HA, HB and HC are A', B', C', L, M and N, respectively. Show that the lines LA', MB' and NC' are concurrent and find, in terms of **a**, **b** and **c**, the position vector of their point of concurrence, K. Hence show that OKH is a straight line.

Prove also that the points A', B', C', L, M and N all lie on a circle with centre K and state the radius of this circle in terms of R. 13

10 A tank with vertical faces has a horizontal rectangular base with sides of lengths X m and $(10 - X)$ m, where X is a continuous random variable having a uniform distribution between 6 and 9. Water is poured into the tank. Given that the volume of water in the tank is $100\ \text{m}^3$, calculate, to three decimal places,

(i) the probability that the depth of water in the tank is greater than 5 m,

(ii) the expected depth of water in the tank.

Also calculate, to the nearest m^3, the least volume of water that the tank must contain for the probability to be at least 0.9 that the depth of water exceeds 5 m.

11 The function f has domain $\mathbb{R}$ and its inverse function is denoted by g.

(i) In the case when

$$f(x) = 3x + |x|,$$

find expressions for $f[f(x)]$ and $g(x)$ in the form $px + q\,|x|$, where p and q are constants.

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show that g exists and state its domain and range. Show, by a graphical method or otherwise, that there is one and only one value of x such that

$$f(x) = g(x).$$

Prove that

$$f'[g(x)]\, g'(x) = 1.$$

Show that $g(2) = 3$, and find the values of $g'(2)$ and $g''(2)$. **20**

Total 11

Joint Matriculation Board

General Certificate of Education

Mathematics
(Pure and Applied Mathematics)
Special Paper

Tuesday 24 June 1986 9.30-12.30

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It is assumed that you have the use of a calculator.

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The use of mathematical tables and slide rules is permitted.

1 Using the substitution $\tan x = u$, or otherwise, find

$$\int \frac{1}{1 + 2\cos^2 x}\,dx.$$

Show that

$$\int_0^{\pi/4} \frac{4x \sin x \cos x}{(1 + 2\cos^2 x)^2}\,dx = \frac{\pi}{72}(9 - 4\sqrt{3}).$$

2 In a multiple choice test paper consisting of 20 questions, an examinee has to choose which *one* of five listed answers to each question is correct. Each correct answer earns 4 marks and 1 mark is deducted for each incorrect answer.

(a) One examinee chooses an answer to each question at random from those listed. Find the probability, to three decimal places, that this examinee's total mark on the paper will be 10 or more.

(b) Another examinee is able to identify three of the listed answers in each of n of the questions as being incorrect, and for each of these questions she chooses one of the remaining two answers at random. For each of the remaining $(20 - n)$ questions she is unable to identify any of the listed answers as being incorrect and chooses her answer at random from the five listed. Find the smallest value of n for which the mean total mark that this examinee can obtain is 20 or more. **9**

3 The polynomial $P(x)$ of degree n is defined by

$$P(x) = \sum_{r=0}^{n} a_r x^r.$$

The exact value of $\int_0^{2h} P(x)\,dx$ is denoted by I. The value of this integral obtained by using Simpson's rule with two strips of width h is denoted by I_s. Show that, when $n \leqslant 3$, $I_s = I$.

Show also that, for $n > 3$,

$$I_s - I = \frac{1}{3}\sum_{r=4}^{n}\left[4 + \frac{(r-5)2^r}{r+1}\right]a_r h^{r+1}.$$ **9**

4 The equations of three planes Π_1, Π_2, Π_3 are

$$\mathbf{r}.\mathbf{n}_1 = p_1, \qquad \mathbf{r}.\mathbf{n}_2 = p_2, \qquad \mathbf{r}.\mathbf{n}_3 = p_3,$$

respectively. No pair of the planes are either parallel or perpendicular to each other. The lines of intersection of Π_2 and Π_3, Π_3 and Π_1, Π_1 and Π_2 are L_1, L_2, L_3, respectively. Show that, for any value of λ, the equation

$$(\mathbf{r}.\mathbf{n}_2 - p_2) + \lambda(\mathbf{r}.\mathbf{n}_3 - p_3) = 0$$

represents a plane containing L_1. Deduce that the equation of the plane Π_4 containing L_1 and perpendicular to Π_1 is

$$\mathbf{r}.[(\mathbf{n}_3.\mathbf{n}_1)\mathbf{n}_2 - (\mathbf{n}_1.\mathbf{n}_2)\mathbf{n}_3] = (\mathbf{n}_3.\mathbf{n}_1)p_2 - (\mathbf{n}_1.\mathbf{n}_2)p_3.$$

Write down the equation of the plane Π_5 containing L_2 and perpendicular to Π_2, and the equation of the plane Π_6 containing L_3 and perpendicular to Π_3.

Show that Π_4, Π_5, Π_6 intersect in a line. **10**

5 The three distinct complex numbers a, b, c are represented in an Argand diagram by the points A, B, C, respectively. The point P represents a complex number z which varies in such a way that

$$\frac{z-a}{z-b}$$

is always purely imaginary and $z \neq a$, $z \neq b$. Show that P moves on the circle whose diameter is AB. Sketch diagrams showing the relationship between A, B, C and the circle on which P moves

(i) in the case when

$$2c = a + b,$$

(ii) in the case when

$$a^2 + b^2 + 2c^2 = 2ac + 2bc.$$

10

6 Two particles, each of mass $2m$, are attached to the ends A and B of a light inextensible string which hangs over a small fixed smooth peg. A light elastic string, of natural length a and modulus of elasticity $2mg$, has one end attached to A and has a particle of mass m attached to the other end C. When the system is in motion with the strings in tension and the parts not in contact with the peg vertical, show that

$$\ddot{x} = \frac{7g}{2} - \frac{5gx}{2a},$$

where x denotes the length of the elastic string at time t.

Given that the length of the elastic string remains constant until B reaches the peg,

(i) determine this length,

(ii) show that the force acting on each particle is constant.

11

7 The function f, whose domain is the set of all real numbers except 0 and 1, is defined by

$$f(x) = \frac{x-1}{x}.$$

Find $f^2(x)$, where f^2 is the composite function ff, and show that $f^3(x) = x$.

The function g is defined in terms of f by

$$g(x) = 1 + (1 - x)g[f(x)].$$

By considering $g[f(x)]$ and $g[f^2(x)]$, find $g(x)$ in terms of x and show that

$$g\left(-\frac{1}{x}\right) = g(x).$$

11

8 A farmer has a flock of N sheep of which 4 are black and the others are white. In a random sample of 3 of the sheep chosen without replacement, let X denote the number that are black.

(i) Find the probability distribution of X.

(ii) Show that the expected value of X is $12/N$.

(iii) Determine the set of values of N for which $X = 1$ is more probable than any of the other possible values of X.

12

9 One end of a light inelastic inextensible string of length l is fixed at a point O and a particle of mass m is attached to its other end. The particle is hanging freely at rest and is then projected horizontally so that it begins to move in a vertical circle with centre O. When the particle reaches a point A on this circle, the string slackens and the particle then travels freely under gravity until it meets the circle again at the point B which lies on the diameter through A. Prove that OA makes an angle of 45° with the upward vertical through O.

Determine the components of the momentum of the particle immediately before it reaches B

(i) in the direction OB,

(ii) in a direction perpendicular to OB.

Given that when the string becomes taut the component of momentum in the direction OB is destroyed and the component perpendicular to OB is conserved, show that the speed of the particle immediately after the string becomes taut is

$$\sqrt{\left(\frac{gl}{\sqrt{2}}\right)}.$$

Show that the string will not become slack again. **17**

10 The tangent to the rectangular hyperbola $xy = 1$ at the point $P\left(p, \frac{1}{p}\right)$ intersects the parabola whose parametric equations are

$$x = t^2 - \tfrac{1}{2},\ y = 2t$$

at the points Q and R. The tangents to the parabola at Q and R meet at T. Given that $t = q$ at Q and $t = r$ at R, find the coordinates of T in terms of q and r. Show that, as p varies, T moves on the curve C whose equation is

$$2y + (x + 1)^2 = 0.$$

Show also that there is only one tangent to the rectangular hyperbola which is also a tangent to the curve C, and find the point at which this tangent touches the rectangular hyperbola. **17**

Total 114

Joint Matriculation Board

General Certificate of Education

Mathematics
(Pure Mathematics)
Special Paper

Tuesday 24 June 1986 9.30-12.30

Careless work and untidy work will be penalised.

There are 11 questions on this paper.

There is no restriction on the number of questions which you may attempt nor on the order in which you attempt them. You are not required to answer all the questions.

It is assumed that you have the use of a calculator.

The marks allocated to questions are given in the right-hand margin of the question paper.

A booklet of definitions, formulae and statistical tables is supplied.

The use of mathematical tables and slide rules is permitted.

1 Show that the equation

$$x^3 + x^2 - ax - 1 = 0$$

has three distinct real roots for $a > 1$. **5**

2 Evaluate

$$\lim_{x \to 0} \frac{\tan px - \tanh px}{\tan x - \tanh x}.$$ **5**

3 Show that

$$2\begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ -2x & -2y & -2z \\ x^2 & y^2 & z^2 \end{vmatrix} = -2(y-z)(z-x)(x-y).$$

Express the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} (x-u)^2 & (x-v)^2 & (x-w)^2 \\ (y-u)^2 & (y-v)^2 & (y-w)^2 \\ (z-u)^2 & (z-v)^2 & (z-w)^2 \end{pmatrix},$$

as the product of two 3×3 matrices $\mathbf{M}_1$ and $\mathbf{M}_2$, where $\mathbf{M}_1$ involves only x, y, z and $\mathbf{M}_2$ involves only u, v, w. Hence, or otherwise, express det $\mathbf{M}$ as the product of six linear factors. **7**

4 Using the substitution $\tan x = u$, or otherwise, find

$$\int \frac{1}{1 + 2\cos^2 x}\,dx.$$

Show that

$$\int_0^{\pi/4} \frac{4x \sin x \cos x}{(1 + 2\cos^2 x)^2}\,dx = \frac{\pi}{72}(9 - 4\sqrt{3}).$$ **8**

5 Given that α is one of the non-real fifth roots of unity, find the numerical values of

$(\alpha - \alpha^4)^2 + (\alpha^2 - \alpha^3)^2$ and $(\alpha - \alpha^4)^2(\alpha^2 - \alpha^3)^2$.

Express the roots of the equation

$$z^4 + 5z^2 + 5 = 0$$

in terms of α. **8**

6 The equations of three planes Π_1, Π_2, Π_3 are

$$\mathbf{r}.\mathbf{n}_1 = p_1, \quad \mathbf{r}.\mathbf{n}_2 = p_2, \quad \mathbf{r}.\mathbf{n}_3 = p_3,$$

respectively. No pair of the planes are either parallel or perpendicular to each other. The lines of intersection of Π_2 and Π_3, Π_3 and Π_1, Π_1 and Π_2 are L_1, L_2, L_3, respectively. Show that, for any value of λ, the equation

$$(\mathbf{r}.\mathbf{n}_2 - p_2) + \lambda(\mathbf{r}.\mathbf{n}_3 - p_3) = 0$$

represents a plane containing L_1. Deduce that the equation of the plane Π_4 containing L_1 and perpendicular to Π_1 is

$$\mathbf{r}.[(\mathbf{n}_3.\mathbf{n}_1)\mathbf{n}_2 - (\mathbf{n}_1.\mathbf{n}_2)\mathbf{n}_3] = (\mathbf{n}_3.\mathbf{n}_1)p_2 - (\mathbf{n}_1.\mathbf{n}_2)p_3.$$

Write down the equation of the plane Π_5 containing L_2 and perpendicular to Π_2, and the equation of the plane Π_6 containing L_3 and perpendicular to Π_3.

Show that Π_4, Π_5, Π_6 intersect in a line. **10**

7 The three distinct complex numbers a, b, c are represented in an Argand diagram by the points A, B, C, respectively. The point P represents a complex number z which varies in such a way that

$$\frac{z-a}{z-b}$$

is always purely imaginary and $z \neq a$, $z \neq b$. Show that P moves on the circle whose diameter is AB. Sketch diagrams showing the relationship between A, B, C and the circle on which P moves

(i) in the case when

$$2c = a + b,$$

(ii) in the case when

$$a^2 + b^2 + 2c^2 = 2ac + 2bc.$$ **10**

8 The function f, whose domain is the set of all real numbers except 0 and 1, is defined by

$$\mathrm{f}(x) = \frac{x-1}{x}.$$

Find $\mathrm{f}^2(x)$, where f^2 is the composite function ff, and show that $\mathrm{f}^3(x) = x$.

The function g is defined in terms of f by

$$\mathrm{g}(x) = 1 + (1-x)\mathrm{g}[\mathrm{f}(x)].$$

By considering $\mathrm{g}[\mathrm{f}(x)]$ and $\mathrm{g}[\mathrm{f}^2(x)]$, find $\mathrm{g}(x)$ in terms of x and show that

$$\mathrm{g}\left(-\frac{1}{x}\right) = \mathrm{g}(x).$$ **11**

9 Given that

$$\mathrm{F}(x) = \int_0^{\pi/2} \frac{\sin^2 xt}{\sin^2 t}\,\mathrm{d}t, \quad x \in \mathbb{R},$$

show that, for $x \neq 0$,

$$\mathrm{F}(x+1) - 2\mathrm{F}(x) + \mathrm{F}(x-1) = \frac{\sin \pi x}{x}.$$

Hence show, using induction or otherwise, that for positive integers n, $\mathrm{F}(n) = \frac{1}{2}n\pi$. **14**

10 The tangent to the rectangular hyperbola $xy = 1$ at the point $P\left(p, \frac{1}{p}\right)$ intersects the parabola whose parametric equations are

$$x = t^2 - \tfrac{1}{2},\ y = 2t$$

at the points Q and R. The tangents to the parabola at Q and R meet at T. Given that $t = q$ at Q and $t = r$ at R, find the coordinates of T in terms of q and r. Show that, as p varies, T moves on the curve C whose equation is

$$2y + (x+1)^2 = 0.$$

Show also that there is only one tangent to the rectangular hyperbola which is also a tangent to the curve C, and find the point at which this tangent touches the rectangular hyperbola. **17**

11 A set T of linear transformations of three dimensional space is such that

(i) each point of the x-axis is mapped to itself,
(ii) the line $x = y$, $z = 0$, is mapped onto itself,
(iii) the line $x = y = z$ is mapped onto itself,
(iv) the point (0,1,0) is mapped into a point not in the x–z plane,
(v) the point (0,0,1) is mapped into a point not in the x–y plane.

Let M denote the set of matrices corresponding to members of the set T. Prove that all members of M are of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1+a & b \\ 0 & 0 & 1+a+b \end{pmatrix},$$

where $1 + a \neq 0$, $1 + a + b \neq 0$. Find the inverse of the above matrix and show that this inverse is in M. Show also that the set M is closed under matrix multiplication. **20**

Total 115

Joint Matriculation Board

General Certificate of Education

Mathematics
(Pure Mathematics with Mechanics)
Special Paper

Tuesday 24 June 1986 9.30-12.30

Careless work and untidy work will be penalised.

There are 10 questions on this paper.

There is no restriction on the number of questions which you may attempt nor on the order in which you attempt them. You are not required to answer all the questions.

It is assumed that you have the use of a calculator.

The marks allocated to questions are given in the right-hand margin of the question paper.

A booklet of definitions, formulae and statistical tables is supplied.

The use of mathematical tables and slide rules is permitted.

1 A light spring lying on a rough horizontal table has natural length l and modulus λ. One end of the spring is fastened to a fixed point on the table and the other end to a particle of mass m. The coefficient of friction between the particle and the table is μ whether the particle is at rest or in motion. The spring is compressed until its length is $(l - a)$ and then released. Determine the smallest value of a for which the particle will move.

The spring is compressed until its length is $l\left(1 - \dfrac{5\mu mg}{2\lambda}\right)$. Show that the particle will move and then come to rest in an equilibrium position in which the length of the spring is $l\left(1 + \dfrac{\mu mg}{2\lambda}\right)$. **8**

2 Using the substitution $\tan x = u$, or otherwise, find

$$\int \frac{1}{1 + 2\cos^2 x}\,dx.$$

Show that

$$\int_0^{\pi/4} \frac{4x \sin x \cos x}{(1 + 2\cos^2 x)^2}\,dx = \frac{\pi}{72}(9 - 4\sqrt{3}).$$ **8**

3 The equations of three planes Π_1, Π_2, Π_3 are

$$\mathbf{r}.\mathbf{n}_1 = p_1, \qquad \mathbf{r}.\mathbf{n}_2 = p_2, \qquad \mathbf{r}.\mathbf{n}_3 = p_3,$$

respectively. No pair of the planes are either parallel or perpendicular to each other. The lines of intersection of Π_2 and Π_3, Π_3 and Π_1, Π_1 and Π_2 are L_1, L_2, L_3, respectively. Show that, for any value of λ, the equation

$$(\mathbf{r}.\mathbf{n}_2 - p_2) + \lambda(\mathbf{r}.\mathbf{n}_3 - p_3) = 0$$

represents a plane containing L_1. Deduce that the equation of the plane Π_4 containing L_1 and perpendicular to Π_1 is

$$\mathbf{r}.[(\mathbf{n}_3.\mathbf{n}_1)\mathbf{n}_2 - (\mathbf{n}_1.\mathbf{n}_2)\mathbf{n}_3] = (\mathbf{n}_3.\mathbf{n}_1)p_2 - (\mathbf{n}_1.\mathbf{n}_2)p_3.$$

Write down the equation of the plane Π_5 containing L_2 and perpendicular to Π_2, and the equation of the plane Π_6 containing L_3 and perpendicular to Π_3.

Show that Π_4, Π_5, Π_6 intersect in a line. **10**

4 Two identical uniform circular cylinders, A and B, rest on a rough horizontal plane with their axes parallel and horizontal. They are positioned so that they just fail to make contact with each other along a generator. A third identical cylinder C, whose axis is parallel to the axes of A and B, is supported symmetrically by A and B. The mass centres of the three cylinders lie in a vertical plane perpendicular to their generators. Show that, for equilibrium, the coefficient of friction between A and C must not be less than $(2 - \sqrt{3})$ and that between A and the plane must not be less than $\frac{1}{3}(2 - \sqrt{3})$. **10**

5 The three distinct complex numbers a, b, c are represented in an Argand diagram by the points A, B, C, respectively. The point P represents a complex number z which varies in such a way that

$$\frac{z - a}{z - b}$$

is always purely imaginary and $z \neq a$, $z \neq b$. Show that P moves on the circle whose diameter is AB. Sketch diagrams showing the relationship between A, B, C and the circle on which P moves

(i) in the case when

$$2c = a + b,$$

(ii) in the case when

$$a^2 + b^2 + 2c^2 = 2ac + 2bc.$$ **10**

6 The function f, whose domain is the set of all real numbers except 0 and 1, is defined by

$$f(x) = \frac{x - 1}{x}.$$

Find $f^2(x)$, where f^2 is the composite function ff, and show that $f^3(x) = x$.

The function g is defined in terms of f by

$$g(x) = 1 + (1 - x)g[f(x)].$$

By considering $g[f(x)]$ and $g[f^2(x)]$, find $g(x)$ in terms of x and show that

$$g\left(-\frac{1}{x}\right) = g(x).$$ **11**

7 Two particles, each of mass $2m$, are attached to the ends A and B of a light inextensible string which hangs over a small fixed smooth peg. A light elastic string, of natural length a and modulus of elasticity $2mg$, has one end attached to A and has a particle of mass m attached to the other end C. When the system is in motion with the strings in tension and the parts not in contact with the peg vertical, show that

$$\ddot{x} = \frac{7g}{2} - \frac{5gx}{2a},$$

where x denotes the length of the elastic string at time t.

Given that the length of the elastic string remains constant until B reaches the peg,

(i) determine this length,

(ii) show that the force acting on each particle is constant. **11**

8 Two elastic particles A and B of masses m and M respectively are placed in a smooth horizontal circular groove. The coefficient of restitution between the particles is e, where $eM < m$. At a given instant B is at rest and A is set moving along the groove with speed V. Prove that the arc through which A moves between the first and second collision subtends an angle $\frac{2\pi(m - eM)}{e(m + M)}$ at the centre.

Find, in terms of m, M, V and e, the speed of B immediately after the second collision. **12**

9 One end of a light inelastic inextensible string of length l is fixed at a point O and a particle of mass m is attached to its other end. The particle is hanging freely at rest and is then projected horizontally so that it begins to move in a vertical circle with centre O. When the particle reaches a point A on this circle, the string slackens and the particle then travels freely under gravity until it meets the circle again at the point B which lies on the diameter through A. Prove that OA makes an angle of 45° with the upward vertical through O.

Find the magnitude of the impulse in the string when the particle reaches B and show that immediately after this impulse the particle is moving with speed

$$\sqrt{\left(\frac{gl}{\sqrt{2}}\right)}.$$

Show that the string will not become slack again. **17**

10 The tangent to the rectangular hyperbola $xy = 1$ at the point $P\left(p, \frac{1}{p}\right)$ intersects the parabola whose parametric equations are

$$x = t^2 - \tfrac{1}{2},\ y = 2t$$

at the points Q and R. The tangents to the parabola at Q and R meet at T. Given that $t = q$ at Q and $t = r$ at R, find the coordinates of T in terms of q and r. Show that, as p varies, T moves on the curve C whose equation is

$$2y + (x + 1)^2 = 0.$$

Show also that there is only one tangent to the rectangular hyperbola which is also a tangent to the curve C, and find the point at which this tangent touches the rectangular hyperbola. **17**

Total 114

Joint Matriculation Board

General Certificate of Education

Mathematics
(Pure Mathematics with Statistics)
Special Paper

Tuesday 24 June 1986 9.30-12.30

Careless work and untidy work will be penalised.

There are 10 questions on this paper.

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It is assumed that you have the use of a calculator.

The marks allocated to questions are given in the right-hand margin of the question paper.

A booklet of definitions, formulae and statistical tables is supplied.

The use of mathematical tables and slide rules is permitted.

1 In a certain video game shots are fired at a moving target. Each time the target is hit, its length is reduced by one half. A game ends when a shot fails to hit the target. The length of the target at the start of a game is 20 cm. The number of shots that a particular player will have in a game has a Poisson distribution with mean 4. Find, in centimetres, to two decimal places, the mean and the standard deviation of the lengths of the target at the end of games by this player. **6**

2 Using the substitution $\tan x = u$, or otherwise, find

$$\int \frac{1}{1 + 2\cos^2 x}\,dx.$$

Show that

$$\int_0^{\pi/4} \frac{4x \sin x \cos x}{(1 + 2\cos^2 x)^2}\,dx = \frac{\pi}{72}(9 - 4\sqrt{3}).$$ **8**

3 The equations of three planes Π_1, Π_2, Π_3 are

$$\mathbf{r}.\mathbf{n}_1 = p_1, \qquad \mathbf{r}.\mathbf{n}_2 = p_2, \qquad \mathbf{r}.\mathbf{n}_3 = p_3,$$

respectively. No pair of the planes are either parallel or perpendicular to each other. The lines of intersection of Π_2 and Π_3, Π_3 and Π_1, Π_1 and Π_2 are L_1, L_2, L_3, respectively. Show that, for any value of λ, the equation

$$(\mathbf{r}.\mathbf{n}_2 - p_2) + \lambda(\mathbf{r}.\mathbf{n}_3 - p_3) = 0$$

represents a plane containing L_1. Deduce that the equation of the plane Π_4 containing L_1 and perpendicular to Π_1 is

$$\mathbf{r}.[(\mathbf{n}_3.\mathbf{n}_1)\mathbf{n}_2 - (\mathbf{n}_1.\mathbf{n}_2)\mathbf{n}_3] = (\mathbf{n}_3.\mathbf{n}_1)p_2 - (\mathbf{n}_1.\mathbf{n}_2)p_3.$$

Write down the equation of the plane Π_5 containing L_2 and perpendicular to Π_2, and the equation of the plane Π_6 containing L_3 and perpendicular to Π_3.

Show that Π_4, Π_5, Π_6 intersect in a line. **10**

4 The three distinct complex numbers a, b, c are represented in an Argand diagram by the points A, B, C, respectively. The point P represents a complex number z which varies in such a way that

$$\frac{z - a}{z - b}$$

is always purely imaginary and $z \neq a$, $z \neq b$. Show that P moves on the circle whose diameter is AB. Sketch diagrams showing the relationship between A, B, C and the circle on which P moves

(i) in the case when

$$2c = a + b,$$

(ii) in the case when

$$a^2 + b^2 + 2c^2 = 2ac + 2bc.$$ **10**

5 The function f, whose domain is the set of all real numbers except 0 and 1, is defined by

$$f(x) = \frac{x - 1}{x}.$$

Find $f^2(x)$, where f^2 is the composite function ff, and show that $f^3(x) = x$.

The function g is defined in terms of f by

$$g(x) = 1 + (1 - x)g[f(x)].$$

By considering $g[f(x)]$ and $g[f^2(x)]$, find $g(x)$ in terms of x and show that

$$g\left(-\frac{1}{x}\right) = g(x).$$ **11**

6 A box contains three electric light bulbs of brand A and two of brand B. Bulbs of brand A have life times that are normally distributed with mean 1200 hours and standard deviation 200 hours, while bulbs of brand B have life times that are normally distributed with mean 1400 hours and standard deviation 400 hours. Two bulbs are selected at random without replacement from the box. Calculate, to three significant figures,

(i) the probability that the first bulb selected will have a life time in excess of 1300 hours,

(ii) the probability that the sum of the life times of the two selected bulbs will exceed 3000 hours. **12**

7 Each of five pupils performed 10 independent trials of an experiment to determine the value of a physical constant. The means of the 10 observed values obtained by the five pupils were, respectively,

28.72, 29.01, 28.48, 28.63, 28.76.

It may be assumed that experimentally determined values of the physical constant are independent and normally distributed with mean equal to the true value of the constant and with variance σ^2. Use all five means given above to obtain

(i) an unbiased estimate of the true value of the physical constant,

(ii) an unbiased estimate of the value of σ^2.

Assuming that $\sigma^2 = 0.36$, calculate 95% confidence limits for the true value of the physical constant. **12**

8 A farmer has a flock of N sheep of which 4 are black and the others are white. In a random sample of 3 of the sheep chosen without replacement, let X denote the number that are black.

(i) Find the probability distribution of X.

(ii) Show that the expected value of X is $12/N$.

(iii) Determine the set of values of N for which $X = 1$ is more probable than any of the other possible values of X. **12**

9 In a multiple choice test paper consisting of 20 questions, an examinee has to choose which *one* of five listed answers to each question is correct. Each correct answer earns 4 marks and 1 mark is deducted for each incorrect answer.

(a) One examinee chooses an answer to each question at random from those listed. Find

(i) the mean and the variance of the total mark that this examinee can obtain,

(ii) the probability, to three decimal places, that this examinee's total mark on the paper will be 10 or more.

(b) Another examinee is able to identify three of the listed answers in each of n of the questions as being incorrect, and for each of these questions she chooses one of the remaining two answers at random. For each of the remaining $(20 - n)$ questions she is unable to identify any of the listed answers as being incorrect and chooses her answer at random from the five listed. Find the smallest value of n for which the mean total mark that this examinee can obtain is 20 or more. Find also, in terms of n, the variance of the total mark that this examinee can obtain. **15**

10 The tangent to the rectangular hyperbola $xy = 1$ at the point $P\left(p, \frac{1}{p}\right)$ intersects the parabola whose parametric equations are

$$x = t^2 - \tfrac{1}{2},\ y = 2t$$

at the points Q and R. The tangents to the parabola at Q and R meet at T. Given that $t = q$ at Q and $t = r$ at R, find the coordinates of T in terms of q and r. Show that, as p varies, T moves on the curve C whose equation is

$$2y + (x + 1)^2 = 0.$$

Show also that there is only one tangent to the rectangular hyperbola which is also a tangent to the curve C, and find the point at which this tangent touches the rectangular hyperbola. **17**

Total 113

Joint Matriculation Board

General Certificate of Education

Mathematics
(Pure and Applied Mathematics)
Special Paper

Tuesday 23 June 1987 9.30-12.30

Careless work and untidy work will be penalised.

There are 11 questions on this paper.

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It is assumed that you have the use of a calculator.

The marks allocated to questions are given in the right-hand margin of the question paper.

A booklet of definitions, formulae and statistical tables is supplied.

The use of mathematical tables and slide rules is permitted.

1 A continuous random variable X is uniformly distributed over the interval $(0, 2\theta)$. Let Y denote the largest of n independent observations of X. Show that the distribution function G, of Y, is given by

$$G(y) = \left(\frac{y}{2\theta}\right)^n, \quad 0 \leqslant y \leqslant 2\theta.$$

Deduce the probability density function of Y and the mean of Y. Show that the variance of Y is

$$\frac{4n\theta^2}{(n+2)(n+1)^2}.$$ 6

2 Determine the values of the real number a for which

$$(2a^2 - 1)x^2 + 6ax - 3a^2 + 12$$

is positive for all real values of x. 7

3 Two fixed points A and B lie on a smooth horizontal plane at a distance $a + b$ apart, where $a \geqslant b$. A particle P of mass m is connected to A and B by light elastic strings, each of modulus λ and of natural lengths a and b, respectively. Initially, P is released from rest at the point B. Show that in the ensuing oscillatory motion the distance between the points at which P is momentarily at rest is

$$\left(1 + \sqrt{\frac{b}{a}}\right)b.$$

Find the period of the motion. 8

4 A car of mass M moves with increasing speed along a straight level road with the engine exerting its maximum power. The total resistance to the motion of the car is proportional to its speed and is such that if the motion were to continue indefinitely the speed of the car would approach a constant value U. Find the time taken for the speed of the car to increase from pU to qU, where $0 < p < q < 1$. Hence write down the work done by the engine in overcoming the total resistance when the speed of the car increases from pU to qU and deduce that

$$\ln\frac{1-p^2}{1-q^2} > q^2 - p^2.$$ 8

5 The equation

$$1 + x + \ln x = 0$$

has one real root which is denoted by α. Let x_k and x_{k+1} be successive approximations to α when Newton's method is used to solve this equation. Denoting the error in x_k by λ_k (so that $\lambda_k = x_k - \alpha$), show that

$$x_{k+1} = \frac{x_k}{1 + x_k}\left[1 + x_k - \lambda_k + \ln\left(1 - \frac{\lambda_k}{x_k}\right)\right].$$

Given that $\frac{\lambda_k}{x_k}$ is sufficiently small for its cube and higher powers to be neglected, show that the error in x_{k+1} is given by the following approximation

$$\lambda_{k+1} \approx -\frac{\lambda_k^2}{2x_k(1 + x_k)}.$$ 9

[You may assume that, when y is small,

$$\ln(1-y) \approx -y - \tfrac{1}{2}y^2.]$$

6 A line drawn parallel to the side AB of a triangle OAB meets the sides OA and OB at C and D, respectively, and $CD{:}AB = p{:}1$. The lines AD and BC intersect at E. Relative to O, the position vectors of A and B are **a** and **b**, respectively. Find the position vector of E relative to O in terms of **a**, **b** and p. Hence, or otherwise, prove that the line through O and E bisects AB. 9

7 In a triangle ABC, D is the mid-point of AB and the angle ADC is denoted by θ. Using the sine rule, or otherwise, show that

$$2 \cot \theta = \cot B - \cot A.$$

Using the cosine rule, or otherwise, show also that

$$2 \cot \theta = \frac{a^2 - b^2}{2\Delta},$$

where Δ is the area of the triangle ABC.
Deduce that, if a^2, b^2 and c^2 are in arithmetical progression, so also are $\cot A$, $\cot B$ and $\cot C$. **10**

8 Given that $x \neq 1$ and

$$1 + x + x^2 + x^3 + \ldots + x^{n-1} = \frac{1 - x^n}{1 - x},$$

show that

$$1 + 2x + 3x^2 + \ldots + (n - 1)x^{n-2} = \frac{1 - nx^{n-1} + (n - 1)x^n}{(1 - x)^2}.$$

An electronic device consists of n components, each of which must be of a certain standard for the device to operate properly. The n components of a device are tested one at a time and the device is accepted only if all the components pass the test. If at any stage of the testing a component fails the test, then no more components are tested and the device is rejected. Let X denote the number of components of a device that will be tested in order to reach a decision on whether or not to reject the device. Given that, independently for each component, the probability of a component failing the test is p, $0 < p < 1$, show that

$$\mathrm{P}(X = n) = (1 - p)^{n-1},$$

and find an expression for $\mathrm{P}(X = k)$, $1 \leqslant k \leqslant n - 1$.
Deduce that the mean number of components tested per device is

$$\frac{1 - (1 - p)^n}{p}.$$

Given that the most probable number of the n components that will be tested in a device is 1, show that

$$n > 1 + \frac{\ln p}{\ln(1 - p)}.$$ **11**

9 A water sprinkler of negligible height sprays small particles of water *in all directions* with speed U. When the sprinkler is placed on level ground at a distance x from a vertical wall, the maximum height above the ground reached by the water on the wall is y. Show that

$$x^2 = 4k(k - y),$$

where $k = U^2/2g$.

The sprinkler is now fixed at a point two metres above a horizontal lawn. Taking $g = 9.8 \text{ ms}^{-2}$, and assuming that all the water falls directly on the lawn,

(i) find the area of lawn that is watered in the case when $U = 7 \text{ ms}^{-1}$,

(ii) find the value of U when the area of lawn watered is $140\pi \text{ m}^2$. **15**

10 Sketch the curve whose equation is

$$y = \left| \frac{x}{1 - x} \right|, \quad x \neq 1,$$

and find the area of the region bounded by the curve, the x-axis and the lines $x = -1$ and $x = \frac{1}{2}$.

Sketch also the curve whose equation is

$$y = \ln \frac{x}{1 - x}, \quad 0 < x < 1.$$

The region bounded by the second curve, the x-axis, the y-axis and the line $y = \ln 3$ is rotated once about the y-axis. Show that the volume of the solid generated is

$$\pi (\ln 2 - \tfrac{1}{4}).$$ **16**

11 The line $y = mx + \lambda$, where $m \neq 0$, meets the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the points P and Q. Show that, when λ varies and m remains constant, the mid-point M of the chord PQ moves on a straight line L and find the equation of L.

Given that one of the points of intersection of the line $y = mx$, where $m \neq 0$, and the ellipse is $C(a \cos \theta, b \sin \theta)$, show that one of the points of intersection of the line L and the ellipse is

$$D(-a \sin \theta, b \cos \theta).$$

Find the Cartesian equation of the locus of the point of intersection of the tangents to the ellipse at C and D as m varies. **17**

Total 116

Joint Matriculation Board

General Certificate of Education

Mathematics
(Pure Mathematics)
Special Paper

Tuesday 23 June 1987 9.30-12.30

Careless work and untidy work will be penalised.

There are 11 questions on this paper.

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It is assumed that you have the use of a calculator.

The marks allocated to questions are given in the right-hand margin of the question paper.

A booklet of definitions, formulae and statistical tables is supplied.

The use of mathematical tables and slide rules is permitted.

1 Show that the equation

$$x(x-3)^2 - 4b = 0,$$

has only one real root for x when $b > 1$.

Given that $1 < b < 5$, show that the above real root α is such that $b < \alpha < 5$. **6**

2 Determine the values of the real number a for which

$$(2a^2 - 1)x^2 + 6ax - 3a^2 + 12$$

is positive for all real values of x. **7**

3 Find an expression in its simplest form for

$$\sum_{r=2}^{n} \ln\left(1 - \frac{1}{r^2}\right).$$ **7**

4 Given that α is a complex ninth root of unity, show that either

$$\alpha^3 = 1 \quad \text{or} \quad 1 + \alpha^3 + \alpha^6 = 0.$$

Find the possible values of z for which

$$\begin{vmatrix} z & \alpha^7 & \alpha^2 \\ \alpha^5 & z & \alpha \\ \alpha^4 & \alpha^8 & z \end{vmatrix} = 0.$$ **8**

5 A line drawn parallel to the side AB of a triangle OAB meets the sides OA and OB at C and D, respectively, and $CD{:}AB = p{:}1$. The lines AD and BC intersect at E. Relative to O, the position vectors of A and B are **a** and **b**, respectively. Find the position vector of E relative to O in terms of **a**, **b** and p. Hence, or otherwise, prove that the line through O and E bisects AB. **9**

6 In a triangle ABC, D is the mid-point of AB and the angle ADC is denoted by θ. Using the sine rule, or otherwise, show that

$$2 \cot \theta = \cot B - \cot A.$$

Using the cosine rule, or otherwise, show also that

$$2 \cot \theta = \frac{a^2 - b^2}{2\Delta},$$

where Δ is the area of the triangle ABC.

Deduce that, if a^2, b^2 and c^2 are in arithmetical progression, so also are $\cot A$, $\cot B$ and $\cot C$. **10**

7 Show that

$$\frac{d}{dx}(\operatorname{cosech} x + \coth x)^m = -m \operatorname{cosech} x (\operatorname{cosech} x + \coth x)^m.$$

Given that

$$I_n = \int_{\ln 2}^{a} (\operatorname{cosech} x + \coth x)^n \, dx,$$

where $a > 0$, show, by considering $I_n - I_{n-2}$, that

$$(n-1)(I_n - I_{n-2}) = 2(3)^{n-1} - 2(\operatorname{cosech} a + \coth a)^{n-1}.$$ **12**

8 Show that the volume of a tetrahedron $ABCD$ can be expressed as

$$\tfrac{1}{6}\,|\overrightarrow{CD}.\mathbf{q}|,$$

where $\mathbf{q} = \overrightarrow{AB} \times \overrightarrow{BC}$.

The edges AB and CD of the tetrahedron are of lengths p_1 and p_2 and lie on the lines

$$\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{n}_1, \quad \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{n}_2,$$

respectively, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit vectors. Show that

$$\overrightarrow{AB} \times \overrightarrow{BC} = \pm p_1\,\mathbf{n}_1 \times (\mathbf{a}_2 - \mathbf{a}_1) + \mathbf{b},$$

where $\mathbf{b}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$.

Hence, or otherwise, find the volume of $ABCD$ when

$$p_1 = 3,\ p_2 = 10,$$
$$\mathbf{a}_1 = 3\mathbf{i}+4\mathbf{j}+5\mathbf{k},\ \mathbf{a}_2 = \mathbf{i}+7\mathbf{j}+2\mathbf{k},$$
$$\mathbf{n}_1 = \tfrac{1}{3}(\mathbf{i}+2\mathbf{j}+2\mathbf{k}),\ \mathbf{n}_2 = \tfrac{1}{3}(2\mathbf{i}+\mathbf{j}+2\mathbf{k}).$$ **12**

9 Show that the set of all matrices of the form

$$\mathbf{M} = \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix},$$

where $a,b,c,d \in \mathbb{C}$ and det $\mathbf{M} = 1$, forms a group G under matrix multiplication. (You may assume that matrix multiplication is associative.)

Find the set of elements C of G such that

$$D^{-1}CD = C$$

for every element D of G. Show that, under matrix multiplication, the set of elements C forms a commutative subgroup of G. **12**

10 Sketch the curve whose equation is

$$y = \left|\frac{x}{1-x}\right|, \quad x \neq 1,$$

and find the area of the region bounded by the curve, the x-axis and the lines $x = -1$ and $x = \frac{1}{2}$.

Sketch also the curve whose equation is

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$$D(-a\sin\theta, b\cos\theta).$$

Find the Cartesian equation of the locus of the point of intersection of the tangents to the ellipse at C and D as m varies. **17**

Total 116

Joint Matriculation Board

General Certificate of Education

Mathematics
(Pure Mathematics with Mechanics)
Special Paper

Tuesday 23 June 1987 9.30-12.30

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A booklet of definitions, formulae and statistical tables is supplied.

The use of mathematical tables and slide rules is permitted.

1 Determine the values of the real number a for which

$$(2a^2 - 1)x^2 + 6ax - 3a^2 + 12$$

is positive for all real values of x. **7**

2 Two fixed points A and B lie on a smooth horizontal plane at a distance $a + b$ apart, where $a \geqslant b$. A particle P of mass m is connected to A and B by light elastic strings, each of modulus λ and of natural lengths a and b, respectively. Initially, P is released from rest at the point B. Show that in the ensuing oscillatory motion the distance between the points at which P is momentarily at rest is

$$\left(1 + \sqrt{\frac{b}{a}}\right)b.$$

Find the period of the motion. **8**

3 A car of mass M moves with increasing speed along a straight level road with the engine exerting its maximum power. The total resistance to the motion of the car is proportional to its speed and is such that if the motion were to continue indefinitely the speed of the car would approach a constant value U. Find the time taken for the speed of the car to increase from pU to qU, where $0 < p < q < 1$. Hence write down the work done by the engine in overcoming the total resistance when the speed of the car increases from pU to qU and deduce that

$$\ln\frac{1 - p^2}{1 - q^2} > q^2 - p^2.$$

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4 A line drawn parallel to the side AB of a triangle OAB meets the sides OA and OB at C and D, respectively, and $CD{:}AB = p{:}1$. The lines AD and BC intersect at E. Relative to O, the position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$, respectively. Find the position vector of E relative to O in terms of $\mathbf{a}$, $\mathbf{b}$ and p. Hence, or otherwise, prove that the line through O and E bisects AB. **9**

5 Show that the centre of mass of a uniform hollow hemispherical bowl of radius r is situated at a distance $\frac{1}{2}r$ from the centre of the hemisphere.

A uniform hemispherical bowl of weight W and centre O rests with its curved surface on a fixed smooth horizontal table with a particle P of weight $\frac{W}{2}$ resting on its inside rough surface. The system is in equilibrium and the coefficient of friction between the particle and the bowl is $\frac{1}{2}$. Show that if θ is the angle between OP and the axis of the bowl, then the greatest possible value of θ is $2\tan^{-1}\frac{1}{2}$. **9**

6 In a triangle ABC, D is the mid-point of AB and the angle ADC is denoted by θ. Using the sine rule, or otherwise, show that

$$2\cot\theta = \cot B - \cot A.$$

Using the cosine rule, or otherwise, show also that

$$2\cot\theta = \frac{a^2 - b^2}{2\Delta},$$

where Δ is the area of the triangle ABC.
Deduce that, if a^2, b^2 and c^2 are in arithmetical progression, so also are $\cot A$, $\cot B$ and $\cot C$. **10**

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Find the Cartesian equation of the locus of the point of intersection of the tangents to the ellipse at C and D as m varies.

17

10 Two particles, P and Q, each of mass m, move on a smooth horizontal floor between two parallel vertical walls. The particles are initially placed apart, one in contact with each wall and the line joining them is perpendicular to the walls. The particles P and Q are then **simultaneously** projected directly towards each other with speeds U and nU, respectively. The coefficient of restitution between the particles and between each particle and each wall is e. Find the impulse between P and Q, in terms of m, n, e and U, when they first collide. Find also the range of values of n, in terms of e, for which the particles move in opposite directions immediately after the first collision.

In the case when $n = \frac{3}{2}$, determine the range of values of e for which P and Q collide again before Q returns to its point of projection.

19

Total 118

Joint Matriculation Board

General Certificate of Education

Mathematics
(Pure Mathematics with Statistics)
Special Paper

Tuesday 23 June 1987 9.30-12.30

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$$2 \cot \theta = \cot B - \cot A.$$

Using the cosine rule, or otherwise, show also that

$$2 \cot \theta = \frac{a^2 - b^2}{2\Delta},$$

where Δ is the area of the triangle ABC.
Deduce that, if a^2, b^2 and c^2 are in arithmetical progression, so also are $\cot A$, $\cot B$ and $\cot C$. **10**

4 A continuous random variable X is uniformly distributed over the interval $(0, 2\theta)$. Let Y denote the largest of n independent observations of X. Show that the distribution function, G, of Y is given by

$$G(y) = \left(\frac{y}{2\theta}\right)^n, \quad 0 \leqslant y \leqslant 2\theta.$$

Deduce the probability density function of Y and the mean of Y. Show that the variance of Y is

$$\frac{4n\theta^2}{(n+2)(n+1)^2}.$$

Given that $T = kY$ is an unbiased estimator of θ, find the constant k in terms of n. Also find the smallest n for which the variance of T is less than $0.01\theta^2$. **10**

5 When a drawing pin is thrown onto a flat surface, the probability that it will come to rest with its point upright is p $(0 < p < 1)$. Let X denote the number of times in n throws that the drawing pin comes to rest with its point upright. Assuming a normal approximation to the distribution of X and ignoring the continuity correction, show that the probability that $\frac{X}{n}$ will differ from p by more than $\frac{1}{\sqrt{n}}$ is less than 0.05.

In 400 throws, the drawing pin came to rest with its point upright 150 times. Use the above probabilistic statement about $\frac{X}{n}$ to derive a confidence interval for the value of p.

What can you say about the confidence level of your interval? **11**

6 A manufacturer wishes to test the claim that the average breaking strength of cable of type A is greater than that of cable of type B. In order to do so, measurements are to be made of the breaking strengths of n sample lengths of cable of type A and of n sample lengths of cable of type B. Denoting the means of the breaking strengths of the samples by $\bar{X}_A$ and $\bar{X}_B$, respectively, the claim will be accepted only if $\bar{X}_A - \bar{X}_B > c$ for some suitably chosen value of c. The breaking strengths of cables of each type are known to be normally distributed with standard deviation 12 units. Determine the least value of n and the corresponding value of c for the decision rule to satisfy both of the following requirements.

(1) If the average breaking strengths of the two types of cable are equal, then the probability is to be only 0.01 that the decision rule will lead to the conclusion that the average breaking strength of cable of type A is greater than that of cable of type B.

(2) If the average breaking strength of cable of type A is 20 units more than that of cable of type B, then the probability is to be at least 0.95 that the decision rule will lead to the conclusion that the average breaking strength of cable of type A is greater than that of cable of type B. **11**

7 Given that the random variable X has a Poisson distribution with mean μ, show that, for any positive integer r,

$$\sum_{x=1}^{r} x\mathrm{P}(X = x) = \mu\mathrm{P}(X \leqslant r - 1),$$

and $$\sum_{x=1}^{r} x(x-1)\mathrm{P}(X = x) = \mu^2\mathrm{P}(X \leqslant r - 2).$$

The number of flaws in a roll of cloth has a Poisson distribution with mean 6. Each roll is inspected for flaws. A roll having nine or fewer flaws is passed. If the count of the flaws in a roll reaches ten, the roll is set aside for corrective treatment. Find, to one decimal place in each case, the mean and the standard deviation of the number of flaws *found* in inspected rolls. **12**

8 The two variables x and y are known to be such that, for all x in the interval $0 \leqslant x \leqslant c$,

$$y = \alpha + \beta x,$$

where α and β are unknown constants. Experimental observations of the values of y corresponding to fixed values of x are subject to independent random errors that are normally distributed with mean zero and standard deviation σ. In order to estimate β it is decided to conduct $2n$ experiments with x having the values x_r $(0 \leqslant x_r \leqslant c)$, $r = 1,2,\ldots,2n$. Let y_r denote the observed value of y when $x = x_r$. Write down an expression for b, the least squares estimate of β, and verify that it is unbiased.

The following two sets of $2n$ values of x are proposed for the purpose of estimating β.

Set 1: $x_r = 0,\ r = 1,2,\ldots,n;\ x_r = c,\ r = n+1,n+2,\ldots,2n.$

Set 2: $x_r = \dfrac{(r-1)c}{2n-1},\ r = 1,2,\ldots,2n.$

Show that the variance of the estimate b obtained from Set 2 is

$$\frac{6(2n-1)\sigma^2}{n(2n+1)c^2}.$$

Find an expression for the variance of the estimate b obtained from Set 1. Hence determine which of the two sets of values of x is the better one for estimating β. **15**

9 Sketch the curve whose equation is

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and find the area of the region bounded by the curve, the x-axis and the lines $x = -1$ and $x = \frac{1}{2}$.

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